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On the Minimum 2-wide Diameter of Cycles with Chords*

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Abstract Let k be a positive integer and G be a k-connected simple graph. The k-wide diameter of graph G, $d_k(G)$, is the minimum integer l such that for any two distinct vertices $x, y \in V(G)$, there are k (internally) disjoint paths with lengths at most l between x and y. Let $C(n,t)$ be the resulting graph by adding t edges to cycle C_n. Define $h(n,t) = \min\{d_2(C(n,t))\}$. In this paper, we compute $h(n,t)$ and obtain that $h(n,2) = \lceil \frac{n}{2} \rceil$. Furthermore, we give the bounds for $h(n,t)$ when $t \geqslant 3$.

Keywords Operations research, graph, network, connectivity, wide diameter
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带弦圈的最小 2 宽直径

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摘要 设 k 为正整数，G 是简单 k 连通图. 图 G 的 k 宽直径，$d_k(G)$, 是指最小的整数 l 使得对任意两不同顶点 $x, y \in V(G)$, 都存在 k 条长至多为 l 的内部不交的连接 x 和 y 的路. 用 $C(n,t)$ 表示在圈 C_n 上增加 t 条边所得的图. 定义 $h(n,t) = min\{d_2(C(n,t))\}$. 本文给出了 $h(n,2) = \lceil \frac{n}{2} \rceil$. 而且，给出了当 t 较大时 $h(n,t)$ 的界.

关键词 运筹学，图，网络，最小性，宽直径
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Graph parameters such as connectivity and diameter have been studied extensively due to their intrinsic importance in graph theory, combinatorics, and their relation to (and application in)fault tolerance and transmission delay in communication networks. The advents of VLSI technology and fiber optics material science has enabled us to design massively parallel computers, complicated VLSI systems, and large scale high speed and wide bandwidth communications networks. All these systems increase their reliability and fault tolerance by incorporating two or more disjoint routing paths between any pair of nodes.

Let G be a k-connected simple undirected graph. For two distinct vertices $u, v \in V(G)$, let $\mathcal{P}_k(u,v)$ be a family of k (internally) disjoint paths between u and v, i.e.

$$\mathcal{P}_k(u,v) = \{P_1, P_2, \cdots, P_k\}, \quad |P_1| \leqslant |P_2| \leqslant \cdots \leqslant |P_k|,$$

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where $|P_i|$ denotes the length of path P_i. The k-wide distance (or simply k-distance), $d_k(u,v)$, between u and v is the minimum $|P_k|$ among all $\mathcal{P}_k(u,v)$'s. The k-wide diameter (or simply k-diameter), denoted by $d_k(G)$, of G is defined as the maximum k-wide distance $d_k(u,v)$ over all distinct vertices $u, v \in V(G)$, i.e.

$$d_k(u,v) = \min_{\mathcal{P}_k(u,v)} |P_k|,$$

and

$$d_k(G) = \max\{d_k(u,v):\ u,v \in V(G) \text{ and u} \neq \text{v}\}.$$

Clearly, $d_1(G)$ is exactly the diameter $d(G)$ of graph G.

Let $C(n,t)$ be the resulting graphs by adding t edges to C_n, the cycle on n vertices. Clearly, every graph in $C(n,t)$ is 2-connected. Let $c(n,t) = \min\{d(C(n,t))\}$. Chung and Garey [2] studied bounds for $c(n,t)$.

There is an analogous problem for 2-wide diameters. Let $h(n,t) = \min\{d_2(C(n,t))\}$.

It is clear from the definition that $h(n,0) = n-1, h(n,1) = n-2$, and$h(n,t) \geqslant h(n,t+1)$. When $t \geqslant 2$, it is less clear what $h(n,t)$ is. In fact, as an open problem, Hsu [1] proposed to compute $h(n,t)$.

In this short article, we study $h(n,t)$ for $t \geqslant 2$. First we give the exact value of $h(n,2)$.

Throughout the paper, we will let C_n be a cycle on n vertices labelled by $u_0, u_1, \ldots, u_{n-1}$. Any graph $G \in C(n,2)$ is obtained from C_n by adding two chords. We denote by $[u_i, u_j]$ the shorter path from u_i to u_j along C_n, and call it the u_iu_j-segment.

Theorem 1

$$h(n,2) = \lceil \frac{n}{2} \rceil$$

for $n \geqslant 4$.

Proof To prove $h(n,2) \leqslant \lceil \frac{n}{2} \rceil$, we need a graph $G \in C(n,2)$ such that $d_2(G) \leqslant \lceil \frac{n}{2} \rceil$. Let $s = \lfloor \frac{n}{2} \rfloor$ and $t = \lceil \frac{s}{2} \rceil$. Consider $G \in C(n,2)$ by adding edges u_0u_s, u_tu_{s+t} to C_n.

Note that u_0, u_t, u_s, u_{s+t} partition the vertices of C_n into four segments:

$$[u_0, u_t], [u_{t+1}, u_s], [u_{s+1}, u_{t+s}], [u_{t+s+1}, u_{n-1}].$$

Now we consider the 2-wide distance $d_2(u_i, u_j)$, where $0 \leqslant i < j \leqslant n-1$. We only consider the case when $i \in [0,t]$, and the other cases are similar. In each situation, we will find two disjoint paths P_1 and P_2 between u_i and u_j, and we will show that $|P_i| \leqslant \lceil \frac{n}{2} \rceil$ for $i = 1, 2$.

Case 1: $j \in [0,s]$. Let

$$P_1 : [u_i, u_j], \qquad P_2 : [u_i, u_0] + u_0u_s + [u_s, u_j].$$

Then $|P_1| \leqslant s$ and $|P_2| = s + 1 - |P_1| \leqslant s$ are two disjoint paths between u_i and u_j.

Case 2: $j \in [s+1, t+s-1]$. Let

$$P_1 : [u_i, u_0] + u_0u_s + [u_s, u_j], \qquad P_2 : [u_i, u_t] + u_tu_{s+t} + [u_{s+t}, u_j].$$

Then

$$2 \leqslant |P_i| \leqslant 2t + 2 - 2 = 2t \leqslant \lceil \frac{n}{2} \rceil, \quad (i = 1, 2)$$

are two disjoint paths between u_i and u_j.

Case 3: $j \in [t+s, n-1]$. Let

$$P_1 : [u_i, u_0] + [u_0, u_j], \qquad P_2 : [u_i, u_t] + u_tu_{s+t} + [u_{s+t}, u_j].$$

Then

$$1 \leqslant |P_1| \leqslant \lceil \frac{n}{2} \rceil, \quad \text{and} \quad |P_2| = n - s + 1 - |P_1| \leqslant n - s \leqslant \lceil \frac{n}{2} \rceil$$

are two disjoint paths between u_i and u_j.

Therefore, $d_2(u_i, u_j) \leqslant \lceil \frac{n}{2} \rceil$ and the upper bound can be reached when $u_i = u_0$ and $u_j = u_{n-1}$. Thus $d_2(G) = \lceil \frac{n}{2} \rceil$ and $h(n, 2) \leqslant \lceil \frac{n}{2} \rceil$.

In the following, we prove that $h(n, 2) \geqslant \lceil \frac{n}{2} \rceil$. Without loss of generality, we suppose that u_0 is one of the endpoints of two chords. We consider three cases according to the way the chords were added.

Case A: The two chords share a common endpoint. Denote the two chords by u_0u_i and u_0u_j with $i < j$. Then

$$d_2(u_1, u_{n-1}) = n - 2 \geqslant \lceil \frac{n}{2} \rceil \quad n \geqslant 2.$$

Case B: The two chords are parallel. Denote the two chords by u_0u_i and u_ju_k with $i < j < k$. If $i \geqslant \lceil \frac{n}{2} \rceil$, $d_2(u_1, u_{n-1}) \geqslant \lceil \frac{n}{2} \rceil$. Otherwise, $d_2(u_{k-1}, u_{k+1}) \geqslant \lceil \frac{n}{2} \rceil$.

Case C: The two chords are crossing. Denote the two chords by u_0u_i and u_ju_k with $j < i < k$. If i or $k - j$ (say i) is more than $\lceil \frac{n}{2} \rceil$ or less than $\lfloor \frac{n}{2} \rfloor$, then

$$d_2(u_{j-1}, u_{j+1}) = i + 1 - 2 = i - 1 \geqslant \lceil \frac{n}{2} \rceil \quad \text{if} \quad i > \lceil \frac{n}{2} \rceil$$
$$d_2(u_{k-1}, u_{k+1}) = n - i + 1 - 2 = n - i - 1 \geqslant \lceil \frac{n}{2} \rceil \quad \text{if} \quad i < \lfloor \frac{n}{2} \rfloor.$$

So assume that $\lfloor \frac{n}{2} \rfloor \leqslant i, k - j \leqslant \lceil \frac{n}{2} \rceil$. Note that

$$d_2(u_0, u_1) = \min\{i, n - (k - j)\},$$
$$d_2(u_0, u_{n-1}) = \min\{n - i, n - (k - j)\},$$
$$d_2(u_i, u_{i-1}) = \min\{i, k - j\}$$
$$d_2(u_i, u_{i+1}) = \min\{n - i, k - j\}.$$

It is easy to check that there is at least one pair of these vertices with 2-distance $\lceil \frac{n}{2} \rceil$ for $i, k - j = \lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$.

Therefore, for any graph $G \in C(n, 2)$, we have $d_2(G) \geqslant \lceil \frac{n}{2} \rceil$. Thus $h(n, 2) \geqslant \lceil \frac{n}{2} \rceil$.

Now we discuss the bounds of $h(n, t)$ when $t \geqslant 3$. In [2], Chung and Garey gave the following upper and lower bounds for $c(n, t)$.

Theorem 2 [2] *If t is even, then*

$$\frac{n}{t+2}-1 \leqslant c(n,t)) \leqslant \frac{n}{t+2}+3.$$

If t is odd, then

$$\frac{n}{t+1}-1 \leqslant c(n,t)) \leqslant \frac{n}{t+1}+3.$$

Since $h(n,t) \geqslant c(n,t)$, the lower bounds of $c(n,t)$ are also the lower bounds for $h(n,t)$. We give upper bounds for $h(n,t)$ here.

Theorem 3 *If t is even, then*

$$\frac{n}{t+2}-1 \leqslant h(n,t) \leqslant 2\lceil\frac{n}{t+2}\rceil+2.$$

If t is odd, then

$$\frac{n}{t+1}-1 \leqslant h(n,t) \leqslant 2\lceil\frac{n}{t+1}\rceil+2.$$

Proof As mentioned above, we just need to prove the upper bounds.

If t is odd, then $t-1$ is even. Then $h(n,t) \leqslant h(n,t-1)$. So we just need to consider the case when t is even.

To get the desired upper bound, we consider a graph $G \in C(n,t)$ constructed in the following way: in C_n, add the t chords v_0v_{2i}, v_1v_{2i+1} for $1 \leqslant i \leqslant t/2$, where $v_0, v_1, \ldots, v_{t+1}$ are vertices along the cycle (in the order) so that the consecutive vertices are at most distance x apart, with

$$x = \lceil\frac{n}{t+2}\rceil.$$

To obtain the 2-diameter of G. Choose any two vertices u and v with $u \in [u_i, u_{i+1}]$ and $v \in [u_j, u_{j+1}]$, where the addition in the subscript is modulo $t+2$. Without loss of generality, assume that $i \leqslant j$ and i is even.

Case 1: u and v lie in a same segment. That is $i=j$. We suppose that $|P(u_i,u)| < |P(u_i,v)|$, where $P(w,z)$ denotes the shortest path between w and z on cycle C_n. Then

$$P_1:[u,v], \qquad P_2:[u,u_i]+u_iu_0+[u_0,u_1]+u_1u_{i+1}+[u_{i+1},v]$$

are two disjoint pathes between u and v with $|P_1| \leqslant x$ and $|P_2| \leqslant 2x+1$ since $|[u,u_i]| + |[u_{i+1},v]| < x$.

Case 2: The two segments containing u and v have a common vertex (one of their endpoints). Then $j=i+1$ or $j=t+1$ and $i=0$. If $j=i+1$, then $j+1=i+2$ is even. Therefore,

$$P_1:[u,v], \qquad P_2:[u,u_i]+u_iu_0+u_0u_{i+2}+[u_{i+2},v]$$

are two disjoint paths between u and v with $|P_1| \leqslant 2x$ and $|P_2| \leqslant x+1+1+x=2x+2$. If $j=t+1$ and $i=0$, then

$$P_1:[u,v], \qquad P_2:[u,u_1]+u_1u_{t+1}+[u_{t+1},v]$$

are two disjoint paths between u and v with $|P_1| \leqslant 2x$ and $|P_2| \leqslant 2x+1$.

Case 3: The two segments containing u and v have no common vertices. Then $[u_i, u_{i+1}] \cap [u_j, u_{j+1}] = \phi$. For convenience, assume that j is even. Thus

$$P_1 : [u, u_i] + u_i u_0 + u_0 u_j + [u_j, v], \qquad P_2 : [u, u_{i+1}] + u_{i+1} u_1 + u_1 u_{j+1} + [u_{j+1}, v]$$

are two disjoint paths between u and v of lengths at most $2x+2$.

Therefore,

$$d_2(G) \leqslant 2x + 2 = 2\lceil \frac{n}{t+2} \rceil + 2.$$

Consequently,

$$h(n,t) \leqslant 2\lceil \frac{n}{t+2} \rceil + 2.$$

Note that if t is even and $t \geqslant n-2$, then $h(n,t) \leqslant 4$. It is easy to construct a graph $C(2n, 2n-2)$ with 2-diameter 4 for $n \geqslant 5$. We conjecture that $h(2n, 2n-2) = 4$ for $n \geqslant 5$.

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